

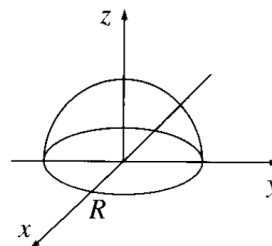
◇ Useful formulas $\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}}$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

1. (20%)

(a) Compute the divergence of the function $\mathbf{v} = r^2 \cos \theta \hat{\mathbf{r}} + r^2 \cos \phi \hat{\boldsymbol{\theta}} - r^2 \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$. (10%)

(b) Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R , resting on the xy plane and centered at the origin (10%).



2. (20%)

(a) Prove that the normal component of $\mathbf{E}$ is discontinuous at any boundary, using Divergence theorem. (6%)

(b) Prove that the tangential component of $\mathbf{E}$ is always continuous, using Stokes' theorem. (6%)

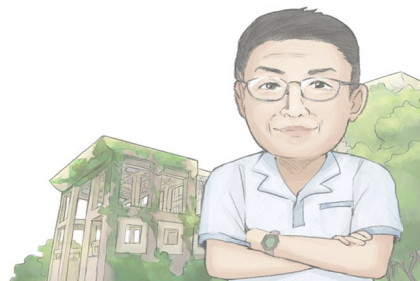
(c) Write down the normal and tangential component of electric fields immediately outside a metal surface with the surface charge density σ . (8%)

3. (20%) The potential of some configuration is given by the expression $V(\mathbf{r}) = A e^{-\lambda r} / r$, where A and λ are constants.

(a) Find the energy density. (6%)

(b) Find the charge density $\rho(\mathbf{r})$. (6%)

(c) Find the total charge Q (do it two different ways) and verify the divergence theorem. (8%)



4. (20%) Consider two concentric spherical shells, of radii a and b ($b > a$). The inner shell is connected to a potential $V(a, \theta)$ (to be given), while the outer shell is grounded $V(b, \theta) = 0$.
- (a) If $V(a, \theta) = V_0$ (constant), find the potential at $r < a$, $a < r < b$, and $r > b$. (10%)
- (b) If $V(a, \theta) = V_0(1 - \cos \theta - \cos^2 \theta)$, find the potential everywhere between the shells ($a < r < b$). (10%) [Hint: use Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$].

5. (20%) A sphere of radius R , centered at the origin, carries charge density

$$\rho(r, \theta) = k \frac{R}{r^2} (R - 2r) \sin \theta,$$

where k is a constant, and r, θ are the usual spherical coordinates.

- (a) Find the monopole, dipole, and quadrupole terms. (10%)
- (b) Find the approximate potential for points on the z axis, far from the sphere. (10%)

[Hint: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{r} \int \rho(\mathbf{r}') d\tau' + \frac{1}{r^2} \int r' \cos \theta' \rho(\mathbf{r}') d\tau' + \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(\mathbf{r}') d\tau' + \dots \right)$].

